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STUDY OF PRACTICAL STABILITY PROBLEMS BY NUMERICAL METHODS AND OPTIMIZATION OF BEAM DYNAMICS*

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The problem of the practical stability (PS) of motion, which generalizes the well-known statement of Chetayev $\{\lambda, A, t_0, T\}$-stability, is considered. For linear systems, criteria for the optimal estimation of PS conditions are obtained. The new concept of directional stability is used to devise algorithms for obtaining extremal sets of stability. The problem of maximizing the PS domain is formulated. The problems of structural parametric optimization of discontinuous dynamic systems, and of maximizing the maximum function with respect to the initial data and the independent variable, are studied. The algorithms of PS and parametric optimization are used to formulate approaches to the optimal design of acceleration and focussing systems. The present paper differs from existing work on the stability of motion in a finite time interval /1, 2/ in that a numerical approach to the study PS is developed on the basis of the results obtained in /3-5/.

1. Numerical studies of practical stability. We consider in the space of the n-dimensional state vector x the sets Φ_t and C_0, which contain the interior points $x(t) \equiv 0$, and the system of equations

$$x^{\cdot} = f(x, t),\ f(0, t) \equiv 0,\ t \in [t_0, T] \tag{1.1}$$

(the dot denotes the derivative with respect to time).

We assuming that the vector function $f(x, t)$ satisfies the conditions of the existence and uniqueness theorem.

Definition 1.1. The unperturbed solution $x(t) \equiv 0$ of system (1.1) is called $\{C_0, \Phi_t, t_0, T\}$-stable if it follows from the initial conditions $x(t_0) \in C_0$ for its trajectories that $x(t) \in \Phi_t,\ t \in [t_0, T]$.

Along with (1.1) we consider the system of differential equations with constantly operating perturbations (Ω_R is the domain of admissible perturbations)

$$x^{\cdot} = f(x, t) + R(x, t),\ \ R(x, t) \in \Omega_R \tag{1.2}$$

Definition 1.2. The unperturbed solution $x(t) \equiv 0$ of system (1.1) is called $\{C_0, \Phi_t, t_0, T, \Omega_R\}$-stable under constantly operating perturbations if $x(t) \in \Phi_t$ $(t \in [t_0, T])$ for any $x(t_0) \in C_0$ and $R(x, t) \in \Omega_R$.

Theorems on PS in the sense of the above definitions are given in /3/. An important point when stating the PS criteria is to prove the existence of Lyapunov functions which satisfy the conditions of the appropriate theorems. Let $C_0 = \{x\colon W(x) < 1\}$, where $W(x)$ is a continuously differentiable positive definite function whose level lines $W(x) = c$ $(0 < c \leqslant 1)$ are closed.

Theorem 1.1. The necessary and sufficient condition for the unperturbed solution $x(t) \equiv 0$ of system (1.1) to be $\{C_0, \Phi_t, t_0, T\}$-stable is that there exist a positive definite Lyapunov function $V(x, t)$ which satisfies the conditions

$$\{x\colon V(x, t) < 1\} \subset \Phi_t,\ \ t \in [t_0, T] \tag{1.3}$$

$$\left(\frac{dV(x,t)}{dt}\right)_{(1.1)} \leqslant 0,\ \ x \in \{x: V(x,t) \leqslant 1\},\ \ t \in [t_0, T] \tag{1.4}$$

$$C_0 \subset \{x\colon V(x, t_0) < 1\} \tag{1.5}$$

The sufficient conditions of the theorem are proved along the lines given in /3/. The necessity is proved with the aid of the function

$$V(x, t) = W(\varphi(x, t, t_0)),\ x(t_0) = \varphi(x(t), t, t_0)$$

This function $V(x, t)$ is positive definite, since $V(0, t) = 0$, $V(x, t) > 0$ for $\|x\| \neq 0$ in

*Prikl.Matem.Mekhan.,51,5,717-723,1987

$[t_0, T]$ because the solution of system (1.1) is unique. Since $V(x,t)$ is constant along trajectories of the system we have $(dV(x,t)/dt)_{(1.1)} = 0$, $t \in [t_0, T]$. We prove inclusion (1.3) by reductio ad absurdum, /5/. Condition (1.5) holds because $\{x: W(x) < 1\} = \{x: V(x, t_0) < 1\}$.

To construct numerically optimal estimates with the aid of the theorem, we consider two classes of sets /3/

$$\Phi_t \triangleq \Gamma_t = \{x: |l_s^*(t)x| \leqslant 1, s = 1, 2, \ldots, N\} \tag{1.6}$$

$$\Phi_t \triangleq \Psi_t = \{x: \psi(x, t) \leqslant 1\} \tag{1.7}$$

and the linear non-stationary system

$$x^{\cdot} = A(t)x + f(t), \quad t \in [t_0, T] \tag{1.8}$$

with any perturbations that satisfy the condition

$$f(t) \in \Omega_R = \{f(t): \|f(t)\| = \Big(\int_{t_0}^{T} \Big(\sum_{i=1}^{n} |f_i(t)|^p\Big)^{p_1/p} dt\Big)^{1/p_1} \leqslant R\} \tag{1.9}$$

We assume that the n-dimensional vectors $l_s(t)$ $(t \in [t_0, T];\ s = 1, 2, \ldots, N)$ are piecewise-continuous functions of t, that the set Ψ_t contains the interior point $x(t) \equiv 0$, and is convex and compact for any $t \in [t_0, T]$; $\Psi_t' = \{x: \psi(x, t) = 1\}$ is the boundary of the set Ψ_t; $\psi(x, t)$ is a continuous function of its arguments along with the partial derivatives with respect to the components of the vector x; the asterisk denotes transposition.

Let $C_0 = \{x: x^*Bx \leqslant c^2\}$, B be a positive definite symmetric matrix.

Criterion 1.1. The necessary and sufficient condition for $\{C_0, \Gamma_t, t_0, T, \Omega_R\}$-stability of system (1.8) is

$$c^2 \leqslant \min_{t \in [t_0, T]} \min_{s=1,2,\ldots,N} \frac{(1 - a_s(t))^2}{l_s^*(t)Q(t)l_s(t)} \tag{1.10}$$

$$a_s(t) < 1, \quad t \in [t_0, T], \quad s = 1, 2, \ldots, N$$

Criterion 1.2. The necessary and sufficient condition for $\{C_0, \Psi_t, t_0, T, \Omega_R\}$-stability of system (1.8) is

$$c^2 \leqslant \min_{t \in [t_0, T]} \min_{\bar{x} \in \Psi_t'} \frac{[g^*(\bar{x}, t)\bar{x} - a_{\bar{x}}(t)]^2}{g^*(\bar{x}, t)Q(t)g(\bar{x}, t)} \tag{1.11}$$

$$g^*(\bar{x}, t)\bar{x} > a_{\bar{x}}(t), \quad \bar{x} \in \Psi_t', \quad t \in [t_0, T]$$

Here, $Q(t)$ is a positive definite symmetric matrix which is the solution of the Cauchy problem

$$Q^{\cdot}(t) = A(t)Q(t) + Q(t)A^*(t), \quad Q(t_0) = B^{-1} \tag{1.12}$$

$$a_{\bar{x}}(t) = R\Big(\int_{t_0}^{t} \Big(\sum_{j=1}^{n} \Big|\sum_{i=1}^{n} x_{ij}(t, \tau) g_i(\bar{x}, t)\Big|^q\Big)^{q_1/q} d\tau\Big)^{1/q_1}$$

$$1/p + 1/q = 1, \quad 1/p_1 + 1/q_1 = 1, \quad g(\bar{x}, t) = \{g_i(\bar{x}, t)\}_{i=1}^{n} = \operatorname{grad}_x \psi(\bar{x}, t)$$

$a_s(t)$ is obtained from $a_{\bar{x}}(t)$ by replacing the vector $g(\bar{x}, t)$ by $l_s(t)$, $\{x_{ij}(t, \tau)\}_{ij=1}^{n} = X(t, \tau)$ are the elements of the fundamental matrix, normalized with respect to τ, corresponding to the homogeneous system (1.8).

Let us state the criterion for stability of the system

$$x^{\cdot} = A(t)(x + f_2(t)) + f_1(t), \quad t \in [t_0, T] \tag{1.13}$$

under perturbations $f_1(t), f_2(t)$ and initial conditions $x(t_0)$ of the domain

$$C_0(t) = \Big\{x(t_0), f_1(t), f_2(t): x^*(t_0)Bx(t_0) + \int_{t_0}^{t} [f_1^*(\tau)C_1(\tau)f_1(\tau) + f_2^*(\tau)C_2(\tau)f_2(\tau)]\,d\tau \leqslant c^2\Big\}$$

where $B, C_1(t), C_2(t)$ are symmetric positive definite matrices.

Criterion 1.3. The necessary and sufficient condition for $\{C_0(t),\ \Gamma_t,\ t_0,\ T\}$-stability of system (1.13) is

$$c^2 \leqslant \min_{t\in[t_0,\,T]}\ \min_{s=1,2,\ldots,N}\ [l_s^*(t)\,Q_1(t)\,l_s(t)]^{-1} \tag{1.14}$$

where the matrix $Q_1(t)$ is the solution of the Cauchy problem

$$Q_1^{\cdot}(t) = A(t)\,Q_1(t) + Q_1(t)\,A^*(t) + C_1^{-1}(t) + A(t)\,C_2^{-1}(t)\,A^*(t),\ Q_1(t_0) = B^{-1} \tag{1.15}$$

A similar criterion holds for a set of type (1.7).

2. Construction of extremal domains of stability and their optimization.

When computing numerically the domains of particle capture in the acceleration mode, the problem arises of finding the entire set of initial conditions under which the trajectories do not leave given sets of phase space. To solve this problem, it is best to introduce the concept of stability with respect to the n-dimensional direction $l\ (\|l\| = 1)$ at the instant $t = t_0$. This directional stability can be defined either in the small (in Lyapunov's sense) or in the finite sense. We shall dwell on the latter, since numerical algorithms and specific estimates will be considered.

Definition 2.1. The unperturbed solution $x(t) \equiv 0$ of system (1.1) is $\{k,\ l,\ \Phi_t,\ t_0,\ T\}$-stable if $x(t) \in \Phi_t\ (t \in [t_0,\ T])$ for any initial conditions $x(t_0) = k_1 l,\ 0 \leqslant k_1 < k$.

This definition is a specific form of the familiar concepts of partial stability and jointly with it, enables constructive approaches to be devised for solving the above problems. For this reason, we shall not dwell on the statement and proof of general theorems on directional stability but merely quote some criteria.

Criterion 2.1. The necessary and sufficient condition for $\{k,\ l,\ \Gamma_t,\ t_0,\ T\}$-stability of system (1.8) is

$$k \leqslant \bar{k}(l) = \min_{t\in[t_0,\,T]}\ \min_{s=1,2,\ldots,N} \frac{1 - |l_s^*(t)\,a(t)|}{|l_s^*(t)\,X(t,t_0)\,l|} \tag{2.1}$$

$$|l_s^*(t)\,a(t)| < 1,\quad s = 1, 2, \ldots, N,\quad t \in [t_0, T]$$

$$\left(a(t) = \int_{t_0}^{t} X(t,\tau)\,f(\tau)\,d\tau\right)$$

Criterion 2.2. The necessary and sufficient condition for $\{k,\ l,\ \Phi_t,\ t_0,\ T\}$-stability of system (1.8) is

$$k \leqslant \bar{k}(l) = \min_{t\in[t_0,\,T]}\ \min_{\bar{x}\in\Psi_t'} \frac{g^*(\bar{x},t)\,(\bar{x} - a(t))}{g^*(\bar{x},t)\,X(t,t_0)\,l} \tag{2.2}$$

$$g^*(\bar{x},t)\,(\bar{x} - a(t)) > 0,\quad g^*(\bar{x},t)\,\bar{x} > 0,\quad \bar{x} \in \Psi_t',\quad t \in [t_0, T]$$

Criterion 2.3. The necessary and sufficient condition for system (1.8) to be $\{k,\ l,\ \Phi_t,\ t_0,\ T\}$-stable in sets (1.6), (1.7) under perturbations that satisfy (1.9) is one of the following:

$$k \leqslant \bar{k}(l) = \min_{t\in[t_0,\,T]}\ \min_{s=1,2,\ldots,N} \left|\frac{1 - a_s(t)}{l_s^*(t)\,X(t,t_0)\,l}\right| \tag{2.3}$$

$$a_s(t) < 1,\ s = 1, 2, \ldots, N,\ t \in [t_0,\ T]$$

$$k \leqslant \bar{k}(l) = \min_{t\in[t_0,\,T]}\ \min_{\bar{x}\in\Psi_t'} \frac{g^*(\bar{x},t)\,\bar{x} - a_{\bar{x}}(t)}{g^*(\bar{x},t)\,X(t,t_0)\,l} \tag{2.4}$$

$$g^*(\bar{x},t)\,\bar{x} > a_{\bar{x}}(t),\quad t \in [t_0, T],\quad \bar{x} \in \Psi_t'$$

The extremal set of stability can be written as

$$C_0^{(\max)} = \{x_0 = k_1 l\colon\ 0 \leqslant k_1 \leqslant \bar{k}(l),\ \forall l\ (\|l\| = 1)\} \tag{2.5}$$

3. Structural parametric optimization of beam dynamics.

These optimization problems are difficult in practice because they are min-max problems, their operation is specified over a long time interval, and to model the trajectories we have to find the acceleration and focussing fields from the relevant Maxwell's equations /3, 6-8/. Here we propose devices for structurally representing the fields in the acceleration and focussing systems, so that the optimization problem can be reduced to a finite form. This means that we can not only simplify the acceleration and focussing system optimization problem, but can perform the optimization in realizable structures.

We know /7, 8/ that the particle velocity in the rectangular accelerating field changes direction on entering and leaving the drift tube. In view of this, we consider the problem of minimizing the quality criterion

$$\min_{\alpha\in C_\alpha} \Phi(x(T)) \tag{3.1}$$

on the trajectories of the system of differential equations

$$\begin{gathered} x^{\cdot} = f^{(i)}(x, t, \alpha), \quad x(t_0) = x(t_0+0) = x_0 \\ t_{i-1} < t < t_i, \quad i = 1, 2, \ldots, N+1 \ (t_{N+1} = T) \end{gathered} \tag{3.2}$$

under the conditions

$$x(t_i+0) = \Phi_i(x(t_i-0), t_i, \alpha), \ i = 1, 2, \ldots, N \tag{3.3}$$

$$t_i = \varphi_i(\alpha), \quad i = 1, 2, \ldots, N \tag{3.4}$$

Here, t_i are the switching points at which the n-dimensional state vector x has the jumps (3.3), $x(t_i+0), x(t_i-0)$ are the values of $x(t)$ to the right and left of the point t_i, $f^{(i)}(x, t, \alpha)$ and $\Phi_i(x, t, \alpha)$ are n-dimensional vector functions, continuous with respect to their arguments along with the partial derivatives with respect to x, α and x, t, α respectively, α is an r-dimensional vector of the optimized parameters, $\varphi_i(\alpha)$ are continuously differentiable functions of α, and C_α is the domain of admissible values of the parameter α.

Problem (3.1) can be solved in accordance with the iterative scheme /9/

$$\alpha^{(i+1)} = P_{C_\alpha}(\alpha^{(i)} - \rho_i C(\alpha^{(i)})), \quad i = 0, 1, 2, \ldots \tag{3.5}$$

where $P_{C_\alpha}(\cdot)$ is the operation of projection onto the set C_α, ρ_i is a sequence of positive numbers that satisfy the convergence condition /9/, $\alpha^{(0)} \in C_\alpha$ is the initial approximation, and the components of the vector gradient $C(\alpha^{(i)})$ are calculated from the relation

$$\begin{aligned} \frac{\partial\Phi}{\partial\alpha_j} = C_j(\alpha^{(i)}) = &-\sum_{s=1}^{N} \psi^*(t_s+0)\left\{\left[\frac{\partial\Phi_s(\bar{x}(t_s-0), t_s, \alpha^{(i)})}{\partial x}\times\right.\right. \\ &f^{(s)}(\bar{x}(t_s-0), t_s, \alpha^{(i)}) - f^{(s+1)}(\bar{x}(t_s+0), t_s, \alpha^{(i)}) + \\ &\left.\left.\frac{\partial\Phi_s(\bar{x}(t_s-0), t_s, \alpha^{(i)})}{\partial t}\right]\frac{\partial\varphi_s(\alpha^{(i)})}{\partial\alpha_j} + \frac{\partial\Phi_s(\bar{x}(t_s-0), t_s, \alpha^{(i)})}{\partial\alpha_j}\right\} - \\ &\sum_{s=1}^{N+1}\int_{t_{s-1}}^{t_s} \psi^*(t)\frac{\partial f^{(s)}(\bar{x}(t), t, \alpha^{(i)})}{\partial\alpha_j}\,dt \\ &j = 1, 2, \ldots, r \end{aligned} \tag{3.6}$$

Here, $\psi(t)$ is a piecewise differentiable vector function of dimensionality n, which is the solution of the boundary value problem

$$\begin{gathered} \psi^{\cdot} = -(\partial f^{(s)}(\bar{x}(t), t, \alpha^{(i)})/\partial x)^*\psi \\ t_{s-1} < t < t_s, \quad s = 1, 2, \ldots, N+1, \quad \psi(T) = -\mathrm{grad}_x\Phi(x(T)) \end{gathered} \tag{3.7}$$

with discontinuities at the switching points

$$\psi(t_s-0) = (\partial\Phi_s(\bar{x}(t_s-0), t_s, \alpha^{(i)})/\partial x)^*\psi(t_s+0), \quad s = 1, 2, \ldots, N \tag{3.8}$$

and $\bar{x}(t)$ is the solution of system (3.2) corresponding to the parameter $\alpha^{(i)}$.

Let M_0 be a compact set of the space of the state vector. We consider the optimization problem

$$I(\alpha^{(0)}) = \min_{\alpha\in C_\alpha} I(\alpha), \quad I(\alpha) = \max_{x_0\in M_0}\max_{t\in[t_0, T]}\Phi(x(t, x_0, \alpha)) \tag{3.9}$$

on the trajectories of the system of differential equations

$$x^{\cdot} = f(x, t, \alpha), \quad t\in[t_0, T] \tag{3.10}$$

on the assumption that the vector function $f(x, t, \alpha)$ is continuous with respect to its arguments along with the partial derivatives with respect to the components of the vectors x, α.

The function $I(\alpha)$ is differentiable at any point $\alpha\in C_\alpha$ with respect to any direction provided that the function $\Phi(x(t, x_0, \alpha))$ is differentiable with respect to the components of the vector α in the open set $C_\alpha' \supset C_\alpha$ /10, 11/. If $\alpha^{(0)}$ is the solution of problem (3.9), then the necessary condition for optimality is that the directional derivative of $I(\alpha)$ evaluated at the point $\alpha^{(0)}$ should be non-negative /10, 11/

$$\max_{\alpha\in C_\alpha} \min_{x_0\in \overline{M}_0} \min_{\tau\in M} \min_{l\in\partial\Phi(x(\tau, x_0, \alpha^{(0)}))} \int_{t_0}^{T} \psi^*(t, \tau, x_0, \alpha^{(0)}, l)\times \partial f(x, t, \alpha^{(0)})/\partial\alpha\,(\alpha-\alpha^{(0)}))\,dt \leqslant 0 \tag{3.11}$$

where $\overline{M}_0, M$ are the sets of points at which the maxima with respect to x_0 and t are reached in (3.9), and $\partial\Phi(x(\tau\ x_0, \alpha^{(0)}))$ is the set of subdifferentials at the point $x_0, \tau, \psi(t, \tau, x_0, \alpha^{(0)}, l)$ is the solution of the Cauchy problem

$$\begin{aligned} &\psi^{\cdot} = -(\partial f(x(t, x_0, \alpha^{(0)}), t, \alpha^{(0)})/\partial x)^*\psi \\ &\psi(\tau, \tau, x_0, \alpha^{(0)}, l) = -l \end{aligned} \tag{3.12}$$

To solve the optimization problem (3.9) numerically, iterative methods of type (3.5) are used, where the vector $C(\alpha^{(i)})$ gives the direction of steepest descent.

Let us dwell on the method of constructing the vector $C(\alpha^{(i)})$ at the i-th iteration. To this end, we specify positive numbers $\varepsilon_i^{(1)}$ and $\varepsilon_i^{(2)}$, defining the accuracy of computing the maxima with respect to x_0 and t. We define the sets

$$\begin{aligned} &\overline{M}_0^{(i)} = \{x_0\in M_0: \max_{t\in[t_0, T]} \Phi(x(t, x_0, \alpha^{(i)})) \geqslant I(\alpha^{(i)}) - \varepsilon_i^{(1)}\} \\ &\overline{M}_{x_0}^{(i)} = \{\tau\in[t_0, T]: \Phi(x(\tau, x_0, \alpha^{(i)})) \geqslant \max_{t\in[t_0, T]} \Phi(x(t, x_0, \alpha^{(i)})) - \varepsilon_i^{(2)}\}, \\ &x_0 \in \overline{M}_0^{()} \end{aligned}$$

The sets $\overline{M}_0^{(i)}$, $\overline{M}_{x_0}^{(i)}$ are covered by a dense discrete mesh $x_0^{(k,i)}$, $\tau_{kj}^{(i)}$ $(j = 1, 2, \ldots, n_k^{(i)}, k = 1, 2, \ldots, Q_i)$, by means of which we compute the vector gradients $C^{(k,j)}(\alpha^{(i)})$ $(j = 1, 2, \ldots, n_k^{(i)}, k = 1, 2, \ldots, Q_i)$ from relations of type (3.6). From the vectors $C^{(k,j)}(\alpha^{(i)})$ we construct the convex hull and find the shortest distance from it to the origin. The point thus found gives the vector of shortest descent at the i-th iteration /10/. To analyse the convergence of the iterative process (3.5), the results of /10, 11/ can be used. The same algorithms can be extended without serious modification to systems with variable structure (3.2).

4. Optimal design of accelerating and focussing systems.

We used the above algorithms to optimize charged particle beam dynamics in different accelerating systems: linear accelerators, electron bunchers, and for the optimal design of power extraction systems in multiresonator klystrons etc.

Instead of writing the unwieldy equations of motion of particles in electromagnetic fields, let us dwell on a commonly encountered model

$$x^{\cdot} = f(x, t, \alpha), \quad y = A(x, t, \alpha)\,y, \quad t\in[0, T], \quad x(0)\in M_0 \tag{4.1}$$

where x, y are the vectors of longitudinal and radial coordinates respectively, α is the r-dimensional vector of optimized parameters, defining the field structure and the accelerating system itself, M_0 is the spread of the particles in longitudinal coordinates, and T is the length of the set-up; the vector y has to satisfy the phase constraints

$$y(t)\in\Gamma_t = \{y: |\,l_s^*(t)\,y\,| \leqslant 1,\ s = 1, 2, \ldots, N\} \tag{4.2}$$

We assume that the y estimate of the particle-trapping domain in the acceleration mode is given in the form $C_0 = \{y: y^*By \leqslant c^2\}$, B is a positive definite matrix. We will formulate the problem of maximizing the trapping in the acceleration mode with respect to the radial coordinate.

Using our above criteria of practical stability, the estimate of the trapping domain is given by

$$c^2 \leqslant \min_{t\in[0, T]} \min_{x_0\in M_0} \min_{s=1, 2, \ldots, N} [l_s^*(t)\,Q(t, x_0, \alpha)\,l_s(t)]^{-1} \tag{4.3}$$

Here, the matrix $Q(t, x_0, \alpha)$ is the solution of the Cauchy problem (1.12) under the condition that $A(t) = A(x(t, x_0, \alpha), t)$. We have thus arrived at a solution of the problem of min-max parametric optimization

$$\min_{\alpha\in C_\alpha} \max_{t\in[0, T]} \max_{x_0\in M_0} \max_{s=1, 2, \ldots, N} l_s^*(t)\,Q(t, x_0, \alpha)\,l_s(t) \tag{4.4}$$

on the trajectories of the matrix differential Eq. (1.12).

The optimization problem (4.4), (1.12) is obtained for the case when the structure of the set of initial conditions is given by the matrix B. If B is unknown, the optimization can be performed with respect to its elements. We then have to satisfy the condition that B must be positive definite. A more sensible method is to use our above method of constructing the extremal sets of stability and write a problem of type (4.4) /3/.

To solve problem (4.4) we used iterative algorithms of type (3.5), taking the case when the maxima with respect to t, x_0, α are not unique. When solving this class of problems we found a high rate of convergence of the iterative processes to the point $\alpha^{(0)}$. The modes obtained were mainly analysed for optimality from the physical stand-point. It must be said that the optimal design of acceleration and focussing systems by our method enables the efficiency of such devices to be greatly improved /3/.

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ON A REMARK OF POINCARE*

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Descriptions of non-autonomous mechanical systems by Poincaré's equations /1/ in the Lagrangian and canonical forms are studied. For systems with a Hamiltonian which depends only on the Chetayev variables /2/ and time, the existence of a complete set of linear (non-commuting) first integrals is proved. The required conditions imposed on the kinetic energy and active forces are studied. Explicit relations for evaluating the integrals by quadratures are obtained. The connection of Poincaré's equations with system of hydrodynamic type is noted. The case of the motion of autonomous mechanical systems when the Lagrange function, expressed in velocity parameters, is independent of the coordinates, was mentioned by Poincaré as being of special interest. This case includes the theory of geodesic left-invariant metrics in Lie groups (a "generalized rigid body" /3/). The primary element of its construction is a Lie group (configuration manifold). Every metric which is defined in it and is invariant under the group operations, defines the kinetic energy. In studies not directly connected with Poincaré's remark, the initial object is the mechanical

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